CHAPTER 1
INTRODUCTION AND BASIC DEFINITIONS

1.1 Introduction

In applied mathematics and physics, orthogonal polynomials have an important place. Moreover, geometrically, orthogonal polynomials are the basis of vector spaces and so any member of this vector space can be expanding a series of orthogonal polynomials.

Almost four decades ago, Konhauser found a pair of orthogonal polynomials which satisfy an additional condition, which is a generalization of orthogonality condition. These polynomials are called biorthogonal polynomials. After Konhauser's study, several properties of these polynomials and another biorthogonal polynomials pairs was found.

In this work, general and basic properties of biorthogonal polynomials are given and two types of biorthogonal polynomials which are namely Konhauser polynomials and Jacobi type biorthogonal polynomials are investigated.

In the first chapter, several basic definitions and theorems about orthogonal polynomials theory are given.

In the second chapter, definition and main theorems of biorthogonal polynomials are obtained.

In the third chapter, Konhauser type biorthogonal polynomials are given and several properties of these polynomials like differential equation, recurrence relation are given.

In the fourth chapter, some bilateral generating function families are obtained for Konhauser biorthogonal polynomials. These generating functions have important applications.

In the fifth chapter, another type of biorthogonal polynomial pair which is suggested by the Jacobi polynomials.

1.2 Gamma Function

The definition of a special function which is defined by using an improper integral is given below. This function is called Gamma Function and has several applications in Mathematics and Mathematical Physics.

Definition 1.1 (Rainville, 1965)

The improper integral

$$\int_0^\infty t^{x-1}\, e^{-t} dt \tag{1.1}$$

converges for any $x > 0$. is called "Gamma Function " and is denoted by Γ:

$$\Gamma(x) = \int_0^\infty t^{x-1}\, e^{-t} dt \quad . \tag{1.2}$$

Some basic properties of Gamma function and given without their proofs. (Rainville, 1965)

$$\int_0^\infty t^n e^{-t} dt = n! = \Gamma(n+1)\,, \tag{1.3}$$

where n is a positive integer.

$$n\,\Gamma(n) = \Gamma(n+1)\ , \tag{1.4}$$

and

$$\Gamma(2\mathrm{b})\,\sqrt{\pi} = 2^{1-2b}\,\Gamma(b)\,\Gamma\left(b+\frac{1}{2}\right), \tag{1.5}$$

where Re (b) > 0.

$$\Gamma(2\,b\;+n)\,\sqrt{\pi}\;\;21-2\mathrm{b}-n=\Gamma\left(b+\frac{1}{2}n\right)\,\Gamma\left(b+\frac{1}{2},n+\frac{1}{2}\right)\;, \tag{1.6}$$

where Re (b) > 0 and n is a non-negative.

$$\Gamma(\mathrm{a})=(\mathrm{a})^{n}\;\frac{(n-1)!}{(\mathrm{a})_{n}}, \tag{1.7}$$

where R(a) > 0 and n is non-negative integer.

Definition 1.2 (Askey, 1999)

Let x be a real or complex number and n is a positive number or zero ,

$$(x)_n \;=\; \frac{\Gamma(x+n)}{\Gamma(x)}=x(x+1)\ldots(x+n-1\;)\;, \tag{1.8}$$

$$(x)_0 \;=\; 1\,,$$

$$(x)_1 \;=\; x\,,$$

$$(x)_2 \;=\; x^2+x\,,$$

is known "Pochammer Symbol".

These are some properties of Pochammer symbol.

1.

$$(\mathrm{c}+n)_{\mathrm{k}}=\frac{(c)_{n+\mathrm{k}}}{(c)_n},$$

where c is a real or complex number and n and k are natural numbers.

2.

$$\frac{n!}{(n-\mathrm{k})!}=\frac{(-n)_{\mathrm{k}}}{(-1)^{\mathrm{k}}},$$

where n and k are natural numbers.

3.

$$\frac{(c)_{2k}}{2^{2k}} = \left(\frac{c}{2}\right)_k \left(\frac{c}{2} + \frac{1}{2}\right)_k,$$

where c is a complex number and k is a natural number.

4.

$$\frac{(2k)!}{2^{2k}.\,k!} = \left(\frac{1}{2}\right)_{\mathrm{k}},$$

where k is a natural number .

There is a usefull lemma for Pochammer symbol. Proof of this lemma can be obtain by directly and elementarly. (Rainville, 1965)

Lemma1.1

$$(\alpha)_{2n} \quad = 2^{2n} \left(\frac{\alpha}{2}\right)_n \left(\frac{\alpha+1}{2}\right)_n, \tag{1.9}$$

Proof

$$(\alpha)_{2n} = \quad \alpha \ \ (\alpha+1)\ \ (\alpha+2)\ldots(\alpha+2n-1)$$

$$= 2^{2n} \left(\frac{\alpha}{2}\right)\left(\frac{\alpha+1}{2}\right)\left(\frac{\alpha}{2}+1\right)\left(\frac{\alpha+1}{2}+1\right)\ldots\left(\frac{\alpha}{2}+n-1\right)\left(\frac{\alpha+1}{2}+n-1\right)$$

$$= 2^{2n} \ \left(\frac{\alpha}{2}\right)\left(\frac{\alpha}{2}+1\right)\ldots\left(\frac{\alpha}{2}+n-1\right)\left(\frac{\alpha+1}{2}\right)\left(\frac{\alpha+1}{2}+1\right)\ldots\left(\frac{\alpha+1}{2}+n-1\right)$$

$$= 2^{2n} \left(\frac{\alpha}{2}\right)_n \left(\frac{\alpha+1}{2}\right)_n .$$

1.3 Orthogonal Polynomials

In this section, definitions and main properties of orthogonal polynomials which are a special case of the biorthogonal polynomials are given. (Askey, 1999)

Definition 1.3

If $w\,(x)$ is a weight function and $p_n(x)$ polynomials are defined over the interval [a ,b] , if

$$\int_{a}^{b} w(x)\, p_n(x)\, p_m(x)\, dx = 0 \;, \qquad m \neq n \quad , \qquad (1.10)$$

is satisfied, then the polynomials $p_n(x)$ are called orthogonal with respect to the weight function $w\,(x)$ over the interval (a , b) , m and n are degrees of polynomials .

There is an additional condition for the orthogonal polynomials which makes them orthonormal.

Definition1. 4

If the polynomials $p_n(x)$ are orthogonal with respect to the weight function w(x), over the interval (a,b) and

$$\|p_n(x)\|^2 = \int_{a}^{b} w(x)\, p_n^2(x) dx = 1 \quad , \quad m = n \quad , \qquad (1.11)$$

is satisfied, then the polynomials $p_n(x)$ are called orthonormal .

There is an equivalent condition for the orthogonality relation (1.10) which is given below.

Theorem 1.1 (Askey, 1999)

It is sufficient for the orthogonality of the polynomials on the interval [a,b] with respect to the weight function $w(x)$ to satisfy the condition

$$\int_{a}^{b} w(x)\, \phi_n(x)\, x^i dx = 0 \;, \;\; i = o, 1,2, \ldots, n-1 \;. \qquad (1.12)$$

Here , $\phi_n(x)$ is a polynomial of degree n.

Proof

If the polynomials $\phi_n(x)$ and $\phi_m(x)$ are orthogonal on the interval [a ,b] with respect to $w(x)$ then

$$\int_{a}^{b} w(x)\,\phi_n(x)\,\phi_m(x)\,dx = 0 \;, m \neq n \;. \tag{1.13}$$

x^i, can be written as linear combinations ,

$$x^{\mathrm{i}} = a_0\phi_0(x) + a_1\phi_1(x) + a_2\,\phi_2(x) + \cdots + a_i\phi_i(x) \;= \sum_{\mathrm{m=0}}^{\mathrm{i}} a_m\,\phi_m(x)\,.$$

Substituting this in (1.12) ,

$$\int_{a}^{b} w(x)\,\phi_n(x)\,x^i dx = \int_{a}^{b} w(x)\,\phi_n(x)\,\left\{\sum_{\mathrm{m=0}}^{\mathrm{i}} a_m\,\phi_m(x)\right\}\,dx$$

$$= \sum_{\mathrm{m=0}}^{\mathrm{i}} a_m \int_{a}^{b} w(x)\,\phi_n(x)\phi_m(x)dx = 0\,.$$

for $0 \leq m \leq i$, $\phi_n(x)$ and $\phi_m(x)$ where $0 \leq m < n$. Hence ,

$$\int_{a}^{b} w(x)\,\phi_n(x)\,x^i dx = 0 \quad , i = 0,1,2,\ldots,n-1 \;.$$

Orthogonal polynomials have several important properties. In this section, general definitions of these properties are given and then obtained special form of them for well-known orthogonal polynomial families.

Definition 1.5 (Askey, 1999)

Any polynomial family $\phi_n(x)$, which is orthogonal on the interval [a ,b] with respect to the weight function $w\,(x)$, satisfies the recurence formula

$$\phi_{N+1}(x) - (xA_N + B_N)\phi_N(x) + C_N\,\phi_{N-1}(x) = 0 \;\;. \tag{1.14}$$

Here A_N, B_N and C_N are constants which depend on N.

Definition 1.6 (Askey, 1999)

Rodrigues Formula for orthogonal polynomials are written as

$$\phi_N(x) = A_N \frac{1}{w(x)} \frac{d^n}{dx^n} \left[w(x)u^n(x)\right] \quad , n = \ 0,1,2 \ldots . \qquad (1.15)$$

Here, $\phi_N(x)$ polynomials are orthogonal with respect to the weight function $w(x)$ and $u^n(x)$ is a polynomial of x.

Definition 1.7 (Askey, 1999)

If the two variable function $\mathrm{F}(x,\mathrm{t})$ has a Taylor series as in the form of

$$\mathrm{F}(x,\mathrm{t}) = \sum_{\mathrm{n}=0}^{\mathrm{i}} a_n \, \phi_n(x) \, t^n, \qquad (1.16)$$

with respect to one of its variables , t, then the function $\mathrm{F}(x,\mathrm{t})$ is called the generating function for the polynomials $\{\phi_n(x)\}$.

1.4 Some Special Orthogonal Polynomial Families

Some well-known orthogonal polynomials family which have several applications in applied mathematics is given at this section. These polynomial families have several properties which are common and obtainable for any orthogonal polynomial family.

1.4.1 Laguerre Polynomials (Rainville, 1965)

For $\alpha > -1$, the $L_n^{(\alpha)}(x)$ polynomials**,** which are orthogonal on $0 \leq x < \infty$ with respect to the weight function $w(x) = x^{\alpha}e^{-x}$ and which are known as Laguerre polynomials are given by,

$$\phi_n(x) = L_n^{(\alpha)}(x) = \sum_{k=0}^{n} (-1)^k \binom{n+\alpha}{n-k} \frac{x^k}{k!} \quad , \qquad n = 0,1,2, \ldots \ . \qquad (1.17)$$

The special case $\alpha = 0$ is $L_n^{(\alpha)}(x) = L_n(x)$. Let we give the first three Laguerre polynomials ,

$$L_0(x) = 1\,, \qquad L_1(x) = 1 - x, \qquad L_2(x) = 1 - 2x + \frac{1}{2}\,x^2,$$

$$L_3(x) = 1 - 3x + \frac{3}{2}x^2 - \frac{1}{6}x^3.$$

Several properties of Laguerre polynomials similar to orthogonal polynomials can be obtained. One of these properties is that it satisfies asecond order diferential equations. Starting from $\frac{d}{dx}\left[x.x^{\alpha}\frac{d}{dx}L_n^{(\alpha)}(x)\right]$, we obtain Laguerre differential equation ,

$$xy'' + (\alpha + 1 - x)y' + ny = 0 \quad . \tag{1.18}$$

where the solutions of this differential equation are Laguerre polynomials can be obtained.

The generating function for the Laguerre polynomials

$$\sum_{n=0}^{\infty} L_n^{(\alpha)}(x)\,t^n = \frac{1}{(1-t)} exp\left(\frac{-xt}{(1-t)}\right). \tag{1.19}$$

can be written. For obtaining the $\left\|L_n^{(\alpha)}(x)\right\|$ norm of Laguerre polynomials, the generating function (1.19) is rewritten as in the form of

$$\sum_{m=0}^{\infty} e^{-x}L_m^{(\alpha)}(x)\,t^m = e^{-x}\,\frac{1}{(1-t)} exp\left(\frac{-xt}{(1-t)}\right), \tag{1.20}$$

by multiplying both sides of (1.19) by $w\,(x) = e^{-x}$ where $m \neq n$. If (1.19) and (1.20) are multiplied side by side and integrate over the interval $(0, \infty)$

$$\sum_{n,m=0}^{\infty}\left[\int_o^{\infty} e^{-x}L_n^{(\alpha)}(x)L_m^{(\alpha)}(x)dx\right]t^{n+m} = \frac{1}{(1-t)^2}\int_o^{\infty} exp\left(\frac{-xt}{(1-t)}\right)dx\,,$$

is obtained. If left hand side of the last equation is seperated for $m = n$ and $m \neq n$, and take the integral at right hand side,

$$\sum_{n,m=0}^{\infty}\left[\int_o^{\infty} e^{-x}L_n^2(x)\,dx\right]t^{2n} + \sum_{n,m=0}^{\infty}\left[\int_o^{\infty} e^{-x}L_n^{(\alpha)}(x)L_m^{(\alpha)}(x)dx\right]\,t^{n+m}$$

$$= \frac{1}{(1-t)^2} . \frac{1-t}{(1+t)}$$

$$= \frac{1}{(1-t^2)} ,$$

is obtained. By using the orthogonality of Laguerre polynomials, for $m = n$, second integral at the left hand side is equal to zero.

If the Taylor series ,

$$\frac{1}{(1-t)} = \sum_{n=0}^{\infty} t^n .$$

is used on the right hand side of the last equality, then

$$\sum_{n,m=0}^{\infty} \left[\int_{o}^{\infty} e^{-x} L_n^2(x)\, dx \right] t^{2n} = \sum_{n=0}^{\infty} t^{2n},$$

is obtained. Thus, equality of the coeffcient of t^{2n} in both sides gives the norm of Laguerre polynomials as

$$\left\| L_n^{(\alpha)}(x) \right\|^2 = \int_{o}^{\infty} e^{-x} L_n^2(x)\, dx = 1 \ . \qquad (1.21)$$

Finally, the recurrence relation for Laguerre polynomial $L_n^{(\alpha)}(x)$ is given as ,

$$(n+1)\, L_{n+1}^{(\alpha)}(x) + (x - 2n - 1 - \alpha)\, L_n^{(\alpha)}(x) + (n + \alpha)\, L_{n-1}^{(\alpha)}(x) = 0 \qquad (1.22)$$

1.4.2 Jacobi Polynomials (Askey, 1999)

For $\alpha > -1$, $\beta > -1$, the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, which is orthogonal on the interval $-1 \le x \le 1$ with respect to the weight function $w(x) = (1-x)^{\alpha}(1+x)^{\beta}$, are given by the formula

$$P_n^{(\alpha,\beta)}(x) = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n+\alpha}{k} \binom{n+\beta}{n-k} (x+1)^{k} (x-1)^{n-k}, n = 1,2,\ldots \ . \qquad (1.23)$$

If α = β, the polynomials $P_n^{(\alpha,\beta)}(x)$ are called " ultraspherical polynomials".

Some special cases of Jacobi polynomials which depend on the values of α and β are given below:

1. For α= β = $-\frac{1}{2}$, the polynomials

$$P_n^{\left(-\frac{1}{2},-\frac{1}{2}\right)}(x) = \sum_{k=0}^{[n/2]} \frac{n!\, x^{n-2k}(x^2-1)^k}{(2k)!\,(n-2k)!} = T_n(x) \ , \tag{1.24}$$

are called “I. Type Chebyshev Polynomials”.

Some of the polynomials $T_n(x)$, are

$$T_0(x) = 1 \ ,$$

$$T_1(x) = x \ ,$$

$$T_3(x) = 2x^2 - 1,$$

$$T_3(x) = 4x^3 - 3x,$$

$$T_4(x) = 8x^4 - 8x^2 + 1 \ .$$

2. For α = β = 0, the polynomials

$$P_n^{(0,0)}(x) = 2^{-n} \sum_{k=0}^{[n/2]} (-1)^k \binom{n}{k} \binom{2n-2k}{n} x^{n-2k} = P_n(x) \ , \tag{1.25}$$

are called " Legendre Polynomials". Here

$$\left[\frac{n}{2}\right] = \begin{cases} \dfrac{n}{2} & if\ n\ is\ even\ , \\ \dfrac{n-1}{2} & if\ n\ is\ odd. \end{cases}$$

If

$$\frac{d}{dx}\left[(1-x^2)(1-x)^{\alpha}(1+x)^{\beta}\frac{d}{dx}P_n^{(\alpha,\beta)}(x)\right],$$

is used to start, the Jacobi differential equation can be obtained as

$$(1-x^2)y'' + [\beta - \alpha - (\alpha + \beta + 2)x]y' + n(n + \beta + \alpha + 1)y = 0 \ , \tag{1.26}$$

which has the solutions as Jacobi polynomials.

Generating function for the Jacobi polynomials is given as

$$\sum_{n=0}^{\infty} p_n^{(\alpha,\beta)}(x)t^n = \frac{2^{\alpha+\beta}}{\sqrt{1-2tx^{2+}t^2}\left[1-t+\sqrt{1-2tx^{2+}t^2}\right]\left[1+t+\sqrt{1-2tx^{2+}t^2}\right]}.$$

Finally, the recurrence relation for Jacobi polynomials is given as

$2(n+1)(n+\alpha+\beta-1)(2n+\beta+\alpha)\, p_{n+1}^{(\alpha,\beta)}(x) - [(2n+\alpha+\beta+1)(\alpha^2-\beta^2)(2n+$
$\alpha\beta+\beta)x]\, p_n^{(\alpha,\beta)} + 2(n+\alpha)(\alpha+\beta)(2n+\alpha+\beta+2)\, p_{n-1}^{(\alpha,\beta)}(x) = 0.$

1.3.3 Hermite Polynomials (Askey, 1999)

The $H_n(x)$ Hermite polynomials, which are orthogonal on the interval $-\infty < x < \infty$ with respect to the weight function $w(x) = e^{-x^2}$ given by,

$$\phi_n(x) = H_n(x) = \sum_{k=0}^{[n/2]} \frac{(-1)^k n!}{k!\,(n-2k)!}(2x)^{n-2k};\ n = 0,1,2,\dots \ . \tag{1.27}$$

Rodrigues Formula for Hermite polynomials is

$$H_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n}(e^{-x^2}) \quad . \tag{1.28}$$

The generating function for the Hermite polynomials is

$$e^{2tx-t^2} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n. \tag{1.29}$$

Norm of the Hermite polynomials is

$$||H_n(x)||^2 = \int_{-\infty}^{\infty} e^{-x^2} H_n(x)\, dx = 2^n\sqrt{\pi}\, n! \ . \tag{1.30}$$

From the equation

$$\frac{d}{dx}\left[e^{-x^2}\ \frac{d}{dx}H_n(x)\right],$$

the Hermite differential equation can be obtained as

$$y'' - 2xy' + 2ny = 0\,, \tag{1.31}$$

which has the solutions as Hermite polynomials.

Finally, the recurrence relation for the Hermite polynomials is given as

$$H_{n+1}(x) - 2xH_n(x) + 2nH_{n-1}(x) = 0\,. \tag{1.32}$$

By using generating function, (1.29),we can obtain the recurrence relation above by following steps.

Take the derivative of both sides in (1.29) with respect to t.

$$(2x-2t)e^{2xt-t^2} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} n\, t^{n-1}$$

$$(2x-2t) \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} = \sum_{n=1}^{\infty} \frac{H_n(x)}{(n-1)!} t^{n-1}$$

$$\sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n - \sum_{\substack{n=0\\ n\to n-1}}^{\infty} \frac{2H_n(x)}{n!} t^{n+1} = \sum_{n=1}^{\infty} \frac{H_n(x)}{(n-1)!} t^{n-1}$$

If the indices are manipulated to make all powers of t as t^n,

$$\sum_{n=0}^{\infty} \frac{2xH_n(x)}{n!} t^n - \sum_{n=1}^{\infty} \frac{2H_{n-1}(x)}{(n-1)!} t^n = \sum_{n=0}^{\infty} \frac{H_{n+1}(x)}{n!} t^n$$

and open some terms to start the summations from 1,

$$2xH_0(x) \ - \sum_{n=1}^{\infty} (2xH_n(x) - 2nH_{n-1}(x))\, \frac{t^n}{n!} = H_0(x) + \sum_{n=1}^{\infty} H_{n+1}(x)\, \frac{t^n}{n!}$$

is obtained. By the equality of the coefficients of the term $\frac{t^n}{n!}$,

$$2xH_n(x) - 2nH_{n-1}(x) = H_{n+1}(x)\,,$$

can be written, which gives the recurrence relation (1.32).

CHAPTER 2

BIORTHOGONAL POLYNOMIALS

In this chapter, basic and alternative definitions of biorthogonal polynomials will be given. First of all, some definitions and notations which are used to give the definition of biorthogonal polynomials are given below.

Definition 2.1

Let *r(x)* and *s(x)* be real polynomials in *x* of degree h $>$ 0 and k $>$ 0 , respectively. Let $R_m(x)$ and $S_n(x)$ denote polynomials of degree *m* and *n* in *r(x)* and *s(x)* , respectively. Then $R_m(x)$ and $S_n(x)$ are polynomials of degree *mh* and *nk* in *x* . Here , the polynomials *r(x)* and *s(x)* are called basic polynomial.

Notation 2.1

Let $[R_m(x)]$ denote the set of polynomials $R_0(x)$, $R_1(x)$, $R_2(x)$,… of degree 0 , 1 , 2 ,…in *r(x)* . Let $[S_n(x)]$ denoted the set of polynomials $S_0(x)$, $S_1(x)$, $S_2(x)$,…of degree 0 , 1 , 2 ,… in *s(x)* .

Definition 2.2 (Konhauser, 1965)

The real-valued function *p(x)* of the real variable *x* is an admissible weight function on the finite or infinite interval (a ,b) if all the moments

$$I_{i,j} = \int_a^b p(x)[r(x)]^i\,[s(x)]^j dx, \qquad i,j = 0,1,2,\dots \tag{2.1}$$

exist, with

$$I_{0,0} = \int_a^b p(x)\,dx \neq 0\,. \tag{2.2}$$

For orthogonal polynomials , it is customary to require *p(x)* be non-negative on the interval (a , b) . This requirement is necessary for the establishment of certain properties for biorthogonal polynomials, this is found necessarily to require that *p(x)* be either nonnegative or nonpositive, with $I_{0,0} \neq 0$, on the interval (a , b) .

By the view of the definitions and notations above, now we can give the definition of biorthogonal polynomials .

Definition 2.3 (Konhauser, 1965)

The polynomial sets $R_m(x)$ and $S_n(x)$ are biorthogonal over the interval (a , b) with respect to the admissible weight function *p(x)* and the basic polynomials *r(x)* and *s(x)* provided the orthogonality conditions

$$J_{m,n} = \int_a^b p(x) R_m(x) S_n(x) dx = \begin{cases} 0 & , \quad m \neq n \\ \neq 0 & , \quad m = n \end{cases}, \quad m,n = 0,1,2,3 \dots , \qquad (2.3)$$

are satisfied .

The orthogonality conditions (2.3) are analogous to the requirements (1.9) for the orthogonality of a single set of polynomials. Following (1.9) , it was pointed out that the requirement that the different from $m = n$ was redundant. The requirement in (1.9) that $J_{m,n}$ be different from zero is not redundant. Polynomial sets $[R_m(x)]$ and $[S_n(x)]$ exist such that

$$J_{m,n} = \begin{cases} 0 & , \quad m \neq n \\ \neq 0 & , \quad m = n \end{cases}, \quad m,n = 0,1,2, \dots , \qquad (2.4)$$

and $J_{k,k} = 0$.

Definition 2.4

If the leading coefficient of polynomial is unity, the polynomial is called monic.

Now, let give the alternative definition for biorthogonality conditions. The following theorem is the analogue of the Theorem(1.12) which gives an alternative definition for orthogonality condition.

Theorem 2.1 (Konhauser, 1965)

If $p(x)$ is an admissible weight function over the interval (a , b) and if the basic polynomials $r(x)$ and $s(x)$ are such that for n = 0, 1, 2…,

$$\int_a^b p(x)[r(x)]^j \, S_n(x)dx = \begin{cases} 0\,, & j = 0,1,2,\dots,n-1, \\ \neq 0\,, & j = n, \end{cases} \tag{2.5}$$

and

$$\int_a^b p(x)[s(x)]^j \, R_m(x)dx = \begin{cases} 0\,, & j = 0,1,2,\dots,m-1, \\ \neq 0\,, & j = m\,, \end{cases} \tag{2.6}$$

are satisfied, then

$$\int_a^b p(x)R_m\,(x)S_n(x)dx = \begin{cases} 0 & , \quad m \neq n \\ \neq 0 & , \quad m = n \end{cases}, \quad m,n = 0,1,2,\dots, \tag{2.7}$$

holds. Conversely, when (2.7) holds then both (2.5) and (2.6) hold.

Proof

If (2.5) and (2.6) hold , then constants, $c_{m,j}$, j = 0 , 1, …, m , ($c_{m,m} \neq 0$) , exist such that

$$R_m(x) = \sum_{j=0}^{m} c_{m,j}\,[r(x)]^j\,. \tag{2.8}$$

If $m \leq n$, then

$$\int_a^b p(x)R_m\,(x)S_n(x)dx = \int_a^b p(x)\sum_{j=0}^{m} c_{m,j}\,[r(x)]^j S_n(x)dx$$

$$= \sum_{j=0}^{m} c_{m,j} \int_a^b p(x)[r(x)]^j \, S_n(x)dx\,.$$

In virtue of (2.5) ,

$$\int_a^b p(x)[r(x)]^j \, S_n(x)dx$$

vanishes except when $j = n = m$.

If $m > n$, then constants $d_{n,j}$, j = 0,1, …, n , ($d_{n,n} \neq 0$) , exist such that

$$S_n(x) = \sum_{j=0}^{m} d_{n,j}\, [s(x)]^j ,$$

and the argument is completed as in the case $m \leq n$.

Now, assume that (2.7) holds. Then constants $e_{m,i}$ and $f_{n,i}$ exist such that

$$[r(x)]^j = \sum_{i=0}^{j} e_{m,i}\; R_i(x) ,$$

and

$$[s(x)]^j = \sum_{i=0}^{j} f_{n,i}\; S_i(x) .$$

If $0 \leq$ j $\leq n$, then

$$\int_a^b p(x)[r(x)]^j\, S_n(x)dx = \int_a^b p(x) \sum_{i=0}^{j} e_{m,i}\; R_i(x) S_n(x)dx$$
$$= \sum_{i=0}^{j} e_{m,i} \int_a^b p(x)\, R_i(x) S_n(x)dx.$$

If i = 1, 2 , …, j , $j < n$, each integral on the right side is zero since (2.7) holds . If $j = n$, the integral on the right side is different from zero. Therefore (2.5) holds. In like manner (2.6) can be established .

CHAPTER 3

BIORTHOGONAL POLYNOMIALS SUGGESTED BY THE LAGUERRE POLYNOMIALS

In this chapter , some well-known biorthogonal polynomials which are generalized form of the Laguerre orthogonal polynomials are going to be given.

First of all , a pair of biorthogonal polynomial family will be given, separately, and then obtain their general properties and definitions. After that, it will be shown that these polynomials are biorthogonal and they satisfy the biorthogonality condition .

3.1 Biorthogonal Polynomial suggested by the Laguerre Polynomials (Konhauser, 1967)

Let $Y_n^c(x;k)$ and $Z_n^c(x;k)$, $n = 0$, 1,…, be polynomials of degree n in *x* and x^k, respectively , where *x* is real, k is a positive integer and c > -1 ,such that

$$\int_0^\infty x^c\, e^{-x}\, Y_n^c(x;k)\, x^{ki}\, dx = \begin{cases} 0 & , \quad for\ i = 0,1,\dots,n-1\,; \\ not\ 0 & , \quad for\ i = n\,; \end{cases} \tag{3.1}$$

and

$$\int_0^\infty x^c\, e^{-x} Z_n^c(x;k) x^i dx = \begin{cases} 0 & , \quad for\ i = 0,1,\dots,n-1\,; \\ not\ 0 & , \quad for\ i = n\,; \end{cases} \tag{3.2}$$

For k = 1, the conditions (3.1) and (3.2) reduce to the orthogonality requirement satisfied by the generalized Laguerre polynomials.

If (3.1) and (3.2) hold, then

$$\int_0^\infty x^c\, e^{-x}\, Y_i^c(x;k)\, Z_n^c(x;k) dx = \begin{cases} 0 & , \quad for\ i = 0,1,\dots,n-1\,; \\ not\ 0 & , \quad for\ i = n\,; \end{cases} \tag{3.3}$$

holds. And conversely, if (3.3) is satisfied then the conditions (3.1) and (3.2) are satisfied by the polynomials $Y_n^c(x;k)$ and $Z_n^c(x;k)$.

For both sets of polynomials, a mixed recurrence relations can be established and the differential equations of order k + 1 can be obtained from these mixed recurrence relations. Pure recurrence relations connecting k + 2 successive polynomials can also be obtained. For k = 1, the recurrence relations and the differential equations for both of polynomial sets reduce to those for the generalized Laguerre polynomials.

Let start with the polynomials $Z_n^c(x;k)$.

3.2 The Polynomial in x^k

One member of the biorthogonal polynomials pair which are suggested by the Laguerrepolynomials is $Z_n^c(x;k)$ and these polynomials are given by the explicit formula

$$Z_n^c(x;k) = \frac{\Gamma(kn+c+1)}{n!}\sum_{j=0}^{n}(-1)^j\binom{n}{J}\frac{x^{kj}}{\Gamma(kj+c+1)} \ , \quad c > -1 \qquad (3.4)$$

which are polynomials of x^k and they are orthogonal with respect to the weight function $x^c\, e^{-x}$ over the interval $(0,\infty)$. These polynomials are reduced to Laguerre polynomials for k =1.

3.2.1 Orthogonality of the Polynomials $Z_n^c(x;k)$

It is known that, the generalized Laguerre polynomials which may be written,

$$L_n^{(\alpha)}(x) = \frac{\Gamma(n+c+1)}{n!}\sum_{j=0}^{n}(-1)^j\binom{n}{J}\frac{x^{j}}{\Gamma(j+c+1)} \ , \quad c > -1 \qquad (3.5)$$

satisfy the orthogonality condition

$$\int_o^\infty x^c\, e^{-x} L_n^c(x;k)\, x^i\, dx = \begin{cases} 0 & , \quad for\ \ i = 0,1,\dots,n-1; \\ not\ 0 & , \quad for\ i = n\,. \end{cases} \qquad (3.6)$$

By using (3.6), the orthogonality of the polynomials $Z_n^c(x;k)$ can be obtained.

In (3.2), replace $Z_n^c(x;k)$ by the right side of (3.4), then carry out the permissible interchange of summation and integration to obtain

$$\frac{\Gamma(kn+c+1)}{n!}\sum_{j=0}^{n}(-1)^j\binom{n}{J}\frac{1}{\Gamma(kj+c+1)}\int_o^\infty e^{-x}\ x^{kj+c+i}dx$$

$$=\frac{\Gamma(kn+c+1)}{n!}\sum_{j=0}^{n}(-1)^j\binom{n}{J}\frac{\Gamma(kj+c+i+1)}{\Gamma(kj+c+1)}$$

$$=\frac{\Gamma(kn+c+1)}{n!}\sum_{j=0}^{n}(-1)^j\binom{n}{J}D^i x^{kj+c+i}|_{x=1}$$

$$=\frac{\Gamma(kn+c+1)}{n!}\ D^i x^{c+i}\sum_{j=0}^{n}(-1)^j\binom{n}{J}x^{kj}|_{x=1}$$

$$=\frac{\Gamma(kn+c+1)}{n!}\ D^i x^{c+i}(1-x^k)^n|_{x=1}\ ,$$

which is zero for i = 0, 1, …, $n-1$, but it is different from zero for i = n.Therefore, the polynomials (3.4) satisfy orthogonality condition (3.2) .

Before determining the other polynomial set of the biorthogonal pair, let obtain several properties satisfied by the polynomials in x^k.

3.2.2 Mixed Recurrence Relations

It is known that, an orthogonal polynomial family has several type of recurrence relations that are consist different terms of polynomials in different orders. The first recurrence relation is

$$xDZ_n^c(x;k)=nkZ_n^c(x;k)-k(kn-k+c+1)_kZ_{n-1}^c(x;k)\ , \tag{3.7}$$

For k = 1, (3.7) reduce to well-known recurrence relation for Laguerre polynomials.

Now, let obtain (3.7) by considering the difference

$$nkZ_n^c(x;k)-k(kn-k+c+1)_kZ_{n-1}^c(x;k)\ , \tag{3.8}$$

If (3.4) is used,

$$\frac{\mathrm{k}\Gamma(kn+c+1)}{(n-1)!}\left[\sum_{j=0}^{n}(-1)^j\binom{n}{J}\frac{x^{kj}}{\Gamma(kj+c+1)}-\sum_{j=0}^{n-1}(-1)^j\binom{n-1}{J}\frac{x^{kj}}{\Gamma(kj+c+1)}\right]$$

$$=\frac{\mathrm{k}\Gamma(kn+c+1)}{(n-1)!}\sum_{j=0}^{n}(-1)^j\left[\binom{n}{J}-\binom{n-1}{J}\right]\frac{x^{kj}}{\Gamma(kj+c+1)},$$

is obtained and may be written

$$\frac{x\Gamma(kn+c+1)}{(n)!}\sum_{j=0}^{n}(-1)^j\binom{n}{J}\frac{kjx^{kj-1}}{\Gamma(kj+c+1)}=xDZ_n^c(x;k)$$

establishing (3.7) .

Alternatively, (3.8) can be written as

$$\frac{k\,x^k\Gamma(k[n-1]+[c+k]+1)}{(n-1)!}\sum_{j=0}^{n}(-1)^j\binom{n-1}{J-1}\frac{x^{k(j-1)}}{\Gamma(k[j-1]+[c+k]+1)}$$

$$=-\frac{k\,x^k\Gamma(k[n-1]+[c+k]+1)}{(n-1)!}\sum_{j=0}^{n-1}(-1)^j\binom{n-1}{J}\frac{x^{kj}}{\Gamma(kj+[c+k]+1)}$$

$$=-k\,x^kZ_{n-1}^{c+k}(x;k)\,,$$

which, together with the preceding result, gives the relation,

$$DZ_n^c(x;k)=-k\,x^{k-1}\,Z_{n-1}^{c+k}(x;k)\,, \tag{3.9}$$

connecting polynomials corresponding to c and c + k. For k = 1, (3.9) also reduce to a well known relation for the generalized Laguerre polynomials.

3.2.3 Differential Equation

Now, let obtain the differential equation which is satisfied by the polynomials $Z_n^c(x;k)$ **.** For obtaining this differential equation, if the difference (3.7) is written as in the form of

$$\frac{xk\Gamma(kn+c+1)}{(n-1)!}\sum_{j=0}^{n}(-1)^j\binom{n-1}{J-1}\frac{x^{kj}}{\Gamma(kj+c+1)},$$

which, in virtue of (3.7), equals $xDZ_n^c(x;k)$. Multiplying by x^c and taking the k th order derivative,

$$\frac{k\Gamma(kn+c+1)}{(n-1)!}\sum_{j=0}^{n}(-1)^j\binom{n-1}{J-1}\frac{x^{k(j-1)+c}}{\Gamma(kj-k+c+1)}$$

$$= -k(kn-k+c+1)_k x^c$$

$$\frac{k\Gamma(kn+c+1)}{(n-1)!}\times\sum_{j=0}^{n-1}(-1)^j\binom{n-1}{J}\frac{x^{kj}}{\Gamma(kj+c+1)}$$

$$= -k(kn-k+c+1)_k\, x^c\, Z_{n-1}^c(x;k)\,.$$

can be obtained. Therefore,

$$D^k[x^{c+1}DZ_n^c(x;k)] = -k(kn-k+c+1)_k x^c Z_{n-1}^c(x;k)\,. \qquad (3.10)$$

is written. Combining (3.7) and (3.10) and eliminating $Z_{n-1}^c(x;k)$, the differential equation of order k + 1,

$$D^k[x^{c+1}DZ_n^c(x;k)] = x^{c+1}DZ_n^c(x;k) - nkx^c Z_n^c(x;k) \qquad (3.11)$$

for the polynomials in x^k, can be obtained.

It is not difficult to verify directly that the polynomials in (3.4) satisfy (3.11). For $k =$ 1, (3.11) reduces to the differential equation for the generalized Laguerre polynomials.

3.2.4 Pure Recurrence Relation

Applying Leibniz's rule for the k th derivative of a product to the left side in (3.10), we get

$$\sum_{i=0}^{k}\binom{k}{i}[D^{k-i}x^{c+1}]\left[D^{i+1}Z_n^c(x;k)\right] = -k\,x^c\,(kn-k+c+1)_k\,Z_{n-1}^c(x;k)\,. \qquad (3.12)$$

The left side is the sum of derivatives of $Z_n^c(x;k)$ from zero through the k +1 order. Elimination of the derivatives by repeated use of (3.7) leads to pure recurrence relations connecting k +2 successive polynomials.

For $k = 1$, (3.12) gives pure recurrence relation for the Generalized Laguerre polynomials.

Now, turn to the polynomials in x which satisfy orthogonality condition (3.1) .

3.3 The Polynomials in x

The polynomials, $Y_n^c(x;k)$, which are the polynomials of x and satisfy the orthogonality condition (3.1) are the other pair of biorthogonal polynomials which are suggested by the Laguerre polynomials.

In this section, some properties of this polynomial family will be obtained and then an explicit formula for them are going to be obtained.

3.3.1 Suggested Recurrence Relation

We seek coefficients $a_{n,j}$ such that the polynomials

$$\sum_{j=0}^{n} a_{n,j}x^{n-j} \qquad (3.13)$$

satisfy the orthogonality condition (3.1). Taking $n =$ 0, 1, 2, 3 and using a method of undermined coefficients, then each to within an arbitrary multiplication constant, the first four polynomials can be obtained as

1,

$x-(c+1),$

$x^2 - (k + 2c + 3)x + (c + 1)(k + c + 1),$

$$x^3 - (3k + 3c + 6)x^2 + [(2k + c + 2)(k + 2c + 3) + (c + 1)(k + c + 1)]x \\ - (c + 1)(k + c + 1)(2k + c + 1)\,.$$

For k =1, the polynomials reduce to the generalized Laguerre polynomials taken to be monic, so in a sense of these polynomials, as well as the polynomials in x^k, may be considered generalizations of the Laguerre polynomials.

The pattern of coefficients suggests the following difference equation for the coefficients

$$a_{n,j} = -[(k + 1)n - j + (-k + c + 1)]a_{n-1,j-1} + a_{n-1,j}\,, \qquad (3.14)$$

where $a_{n,0} = 1$ for all n and $a_{i,j} = 0$ if $i < j$. (3.14) can be used as the basis of a conjecture for a recurrence relation for the polynomials in x. Then, this recurrence relation is used to show that the polynomials satisfying (3.1). By uniqueness, the polynomials which satisfy the recurrence relation, can be sacrificed the monic property of the polynomials by modifying the difference equation (3,14) to

$$knb_{n,j} = [kn + j + (-k + c + 1)]b_{n-1,j} - b_{n-1,j-1}\,, \qquad (3.15)$$

where $b_{0,0} = 1, b_{i,j} = 0$ if $i < j, , b_{i,-1} = 0$ for all i, so the polynomials in x are givenby

$$Y_n^c(x; k) = \sum_{j=0}^{n} b_{n,j} \ \ x^j. \qquad (3.16)$$

$k = 1$, (3.15) is recurrence relation for the coefficients of the generalized Laguerre polynomials.

Substituting for $b_{n,j}$ in (3,16), we get

$$knY_n^c(x; k) = \sum_{j=0}^{n} [kn + j + (-k + c + 1)]\, b_{n-1,j} \ \ x^j - \sum_{j=0}^{n} b_{n-1,j-1} \ \ x^j.$$

Replacing n by $n + 1$, and noting that $b_{n,n+1} = 0$ we have

$$k(n+1)Y_{n+1}^c(x;k) = \sum_{j=0}^{n+1}[kn+j+c+1]\, b_{n,j}\ \ x^j - \sum_{j=0}^{n+1} b_{n,j-1}\ \ x^j$$

$$= (kn+c+1)Y_n^c(x;k) + \sum_{j=0}^{n} j\, b_{n,j}\ \ x^j - \sum_{j=0}^{n} b_{n,j}\ \ x^{j+1}\,.$$

The first sum on the right side is $xDY_n^c(x;k)$ and the second is $xY_n^c(x;k)$, therefore, a suggested recurrence relation for the polynomials in x is

$$k(n+1)Y_{n+1}^c(x;k) = xDY_n^c(x;k) + (kn+c+1-x)Y_n^c(x;k)\ \ . \tag{3.17}$$

3.3.2 Biorthogonality of The Polynomials $Y_n^c(x;k)$

To establish that the polynomials in x that satisfy (3.17), comprise the other set of the biorthogonal pair, it must be shown that (3.1) is satisfied by induction.

For n =0, the integral in (3.1) has the nonzero value $\Gamma(c+1)$ only permissible value of i, namely $i = 0$.

For n =1, the integral in (3.1) is zero for $i = 0$ and nonzero for $i = 1$. For $i = 0$,

$$\int_o^\infty x^c\, e^{-x}\, Y_1^c(x;k)\, dx$$

$$= \int_o^\infty x^c\, e^{-x}\, k^{-1}(c+1-x)\, dx$$

$$= k^{-1}[(\mathrm{c}+1)\Gamma(c+1) - \Gamma(c+2)] = 0\,,$$

is written, where $Y_1^c(x;k) = \ k^{-1}(c+1-x)$ was obtained from (3,17) for $n = 0$.

For $i = 1$,

$$\int_o^\infty x^c\, e^{-x} Y_n^c(x;k) dx$$

$$= \int_{o}^{\infty} x^{c+k}\, e^{-x} k^{-1}(c+1-x)dx$$

$$= k^{-1}[(\mathrm{c}+1)\Gamma(c+k+1) - \Gamma(c+k+2)]$$

$$= -\Gamma(c+k+1) \neq 0\,,$$

can be written. Continuing the induction argument, let assumed that the polynomials $Y_i^c(x;k), i = 0\,,1\,,\ldots,n$, are obtained by repeated application of (3.17), satisfy orthogonality relation (3.1). To complete the induction argument, it must be shown that

$$\int_{o}^{\infty} x^{c}\, e^{-x} Y_{n+1}^{c}(x;k) x^{ik} dx = \begin{cases} 0, & i = 0,1,\ldots,n\,; \\ \neq 0, & i = n-1\,. \end{cases} \tag{3.18}$$

Substituting for $Y_{n+1}^{c}(x;k)$ as given by (3.17) ,

$$k^{-1}(n+1)^{-1} \int_{o}^{\infty} x^{c+ik+1}\, e^{-x}\, DY_n^c(x;k)dx$$

$$+k^{-1}(n+1)^{-1} \int_{o}^{\infty} (kn+c+1-x)\, x^{c+ik}\, e^{-x}\, Y_n^c(x;k)\, dx\,.$$

is obtained and it can be written as

$$(n+1)^{-1} \int_{o}^{\infty} x^{c+ik}\, e^{-x} (n-i) Y_n^c(x;k) dx\ \ . \tag{3.19}$$

By hypothesis, $Y_n^c(x;k)$ is orthogonal to x^{ik}, for $0 \leq i < n$, Therefore, for $i < n$, the integral in (3.19) is zero.For $\mathrm{i} = n+1$, the integral has the value

$$-(n+1)^{-1} \int_{o}^{\infty} x^{c+ik+k}\, e^{-x} Y_n^c(x;k) dx = (-1)^{n+1}\, \Gamma(c+kn+k+1)\,,$$

which is different from zero. The right side is obtained by n applications of (3.17), each followed by an integration by parts.

3.3.3 Expression for $Y_n^c(x;k)$ (Preiser, 1962)

Now, let obtain an explicit formula for the polynomials $Y_n^c(x;k)$. Preiser obtained a closed form for the polynomials in x for the case $k=2$ by applying Cauchy's Theorem to the integral form solution of

$$x\, D^3 Y_n(x;2) + (1+c-3x)\, D^2 Y_n(x;2) + 2(x-1-c)\, DY_n(x;,2) = 2nY_n(x;2). \quad (3.20)$$

A closed form for the polynomials is desirable but is not essential, since certain properties of the polynomials can established without one. By a method similar to that of Preiser, polynomials solutions of (3.20) in integral form can be found. Conjecture the form of the integral, for general case, show that the polynomials so obtained satisfy (3.17) , and then , using the integral form , establish relations which will be used to derive a differential equation for the polynomials. Equation (3.20) may be written

$$x(y''' - 3y'' + 2y') + \; [\,(1+c)\, y'' - 2(1+c)y' - 2ny] = 0 \qquad (3.21)$$

A solution of the form

$$y = \oint_c e^{-xt} \emptyset(t) dt\,, \qquad (3.22)$$

where the function $\emptyset(t)$ and the contour C are to be determined , is assumed . Differentiating successively and substituting (3.21),

$$-x \oint_c (t^3 + 3t^2 + 2\,t) e^{-xt} \emptyset(t) d + \oint_c [(1+c)t^2 + 2(1+c)t - 2n]\, e^{-xt} \emptyset(t) dt = 0\,.$$

is obtained. Integrating the first integral by parts,

$$0 \;= (t^3 + 3t^2 + 2t)\emptyset(t) e^{-xt}|_c$$
$$- \oint_c [(3t^2 + 6t + 2)\emptyset(t) + (t^3 + 3t^2 + 2t)\emptyset'(t)]\, e^{-xt} dt$$

$$+\oint_c [(1+c)t^2 + 2(1+c)t - 2n]\, e^{-xt}\emptyset(t)dt\,,$$

can be obtained. $\emptyset(t)$ is choosen such that

$$[(1+c)t^2 + 2(1+c\,)t - 2n - 3t^2 - 6t - 2]\emptyset(t) - (t^3 + 3t^2 + 2t)\emptyset'(t)] = 0, \quad (3.23)$$

and the contour C such that

$$(t^3 + 3t^2 + 2t)\emptyset(t)e^{-xt}|_C = 0. \quad (3.24)$$

From (3.22),

$$\frac{\emptyset'(t)}{\emptyset(t)} = -\frac{n+1}{t} + \frac{c+2n}{t+1} - \frac{n+1}{t+2}\,,$$

is written. Hence $\emptyset(t) = K\,(t+1)^{c+2n}/t^{n+1}(t+2)^{n+1}$, where K is an arbitrary constant which we shall take equal to $k/2\pi i$.

Substituting into (3.24), we require the contour C is required to be that

$$\frac{k}{2\pi i}\,\frac{(t+1)^{c+2n+1}}{t^2(t+2)^n}\,|_C = 0\,.$$

If C is taken to be a closed contour encircling t = 0 , but not t $= -1$ or t $= -2$, the (3.24) holds and

$$y = \frac{k}{2\pi i}\oint_c \frac{e^{-xt}(t+1)^{c+2n}}{t^{n+1}(t+2)^{n+1}}dt\,.$$

is obtained. On the basis of this integral, we conjecture that the polynomials $Y_n^c(x;k)$ are given by

$$\emptyset_n(x) = \frac{k}{2\pi i}\oint_c \frac{e^{-xt}(t+1)^{c+kn}}{[(t+1)^k - 1]^{n+1}}dt\,. \quad (3.25)$$

In the view of the uniqueness, it sufficies to show that the polynomials (3.25) satisfy recurrence relation (3.16) and we have

$$xD\emptyset_n(x) - x\emptyset_n(x) = -\frac{K}{2\pi i}\oint_c \frac{e^{-xt}(t+1)^{c+kn+1}}{[(t+1)^k - 1]^{n+1}}\,dt \ . \tag{3.26}$$

Integrating by parts and applying (3,25), the right side of (3,26) becomes

$$-(c+kn+1)\emptyset_n(x) + k(n+1)\emptyset_{n+1}(x) \ .$$

Therefore ,

$$xD\emptyset_n(x) - x\emptyset_n(x) = -(c+kn+1)\emptyset_n(x) + k(n+1)\emptyset_{n+1}(x) \ ,$$

which is (3.17) with $\emptyset_n(x)$ in place of $Y_n^c(x;k)$ in summary,

$$Y_n^c(x;k) = \frac{k}{2\pi i}\oint_c \frac{e^{-xt}(t+1)^{c+kn}}{[(t+1)^k - 1]^{n+1}}\,dt \tag{3.27}$$

is obtained. Applying Cauchy's theorem to (3.27), we obtain the following representation for the polynomials in x:

$$Y_n^c(x;k) = \frac{k}{n!}\frac{\partial^n}{\partial t^n}\left[\frac{e^{-xt}(t+1)^{c+kn}}{(t^{k-1} + kt^{k-2} + \cdots + k)^{n+1}}\right]\Big|_{t=0} \ .$$

3.4 The Integrals $J_{n,n}$

Now for the biorthogonal polynomials $Z_n^c(x;k)$ and $Y_n^c(x;k)$, evaluation of the integral

$$J_{n.n} = \int_o^\infty x^c\, e^{-x} Y_n^c(x;k) Z_n^c(x;k)\,dx \ .$$

which is the biorthogonality condition of polynomials as suggested by the Laguerre polynomials will be obtained. First, show that $b_{n,n} = (-1)^n/k^n n!$, $n = 0$, 1 , 2 ,… will be shown by induction.

For $n = 0$, $b_{0,0} = 1$.

For $n = 1$, from (3.16) $b_{1,1} = -1/k$ is obtained .

Let $b_{n-1,n-1} = (-1)^{n-1}/k^{n-1}(n-1)!$. Taking j $= n$ in (3.15) , and pointed out that $b_{n-1,n} = 0$,

$$b_{n,n} = -b_{n-1,n-1}/kn = (-1)^n/k^n n! \,,$$

is obtained which complete the induction argument .

In virtue of the orthogonality of x^j and $Z_n^c(x;k)$ for j < n ,

$$J_{n.n} = \int_o^\infty x^c\, e^{-x} Z_n^c(x;k) \sum_{j=0}^{n} b_{n,j} x^j \, dx$$

$$= \int_o^\infty x^c\, e^{-x} Z_n^c(x;k) b_{n,n}\, x^n dx \,.$$

can be written. Subsisting for $b_{n,n}$ and proceeding as in the establishment of the biorthogonality property of the polynomials in x^k, then

$$J_{n.n} = \frac{(-1)^n}{k^n n!} \frac{\Gamma(kn+c+1)}{n!}\; D^n x^{c+n}(1-x^k)^n|_{x=1}$$

$$= \frac{\Gamma(kn+c+1)}{n!} \quad ,$$

is obtained which , for k = 1 , is the value of the corresponding integral for the generalized Laguerre polynomials.

CHAPTER 4

SOME PROPERTIES OF KONHAUSER BIORTHOGONAL POLYNOMIALS

In this Chapter, several generating functions for both of polynomials $Z_n^c(x;k)$ and $Y_n^c(x;k)$ are going to obtain. These generating functions are bilateral generating functions. (Srivastava 1973 & Srivastava 1980)

4.1 Bilateral Generating Functions for $Z_n^c(x;k)$ Konhauser Polynomials

First, several bilateral generating functions for $Z_n^c(x;k)$ Konhauser Polynomials will be obtained and proved.

Theorem 4.1

The polynomials $Z_n^\alpha(x;k)$ can be expressed as a finite sum of $Z_n^\alpha(y;k)$ in the form

$$Z_n^\alpha(x;k) = \left(\frac{x}{y}\right)^{kn} \sum_{r=o}^{n} \binom{\alpha+kn}{kr} \frac{(kr)!}{r!} \left[(y/x)^k - 1\right]^r Z_{n-r}^\alpha(y;k) \quad . \tag{4.1}$$

The following results will be required in further analysis ,

$$Z_n^\alpha(x) = \frac{\Gamma(\alpha+kn+1)}{n!} \sum_{j=0}^{n} (-1)^j \binom{n}{J} \frac{x^{kj}}{\Gamma(\alpha+kj+1)} \quad . \tag{4.2}$$

And

$$\sum_{n=0}^{\infty} Z_n^\alpha(x;k) \frac{t^n}{(\alpha+1)_{kn}}$$

$$= e^t F_k[-;\ (\alpha+1)/k\,, \dots, (\alpha+1)/k\,; -(x/k)^k t]\,, \tag{4.3}$$

Since k is a positive integer .

Proof

In the generating function (4.3) , if t = $(y/x)^k z$,

$$\sum_{n=0}^{\infty} Z_n^{\alpha}(x;k) \frac{(y/x)^{kn} z^n}{(\alpha+1)_{kn}} = exp\{(y/x)^k z\} \quad {}_0F_k[-(xy/k^2)^k z], \quad (4.4)$$

is obtained which, on interchanging x and y, gives

$$\sum_{n=0}^{\infty} Z_n^{\alpha}(x;k) \frac{(x/k)^{kn} z^n}{(\alpha+1)_{kn}} = exp\{(y/x)^k z\} \quad {}_0F_k[-(xy/k^2)^k z], \quad (4.5)$$

where, for convenience,

$${}_0F_k[\xi] \equiv F_k[-;(\alpha+1)/k, \dots, (\alpha+k)/k;\ \xi].$$

From (4.4) and (4.5) , it follows once that

$$\sum_{n=0}^{\infty} Z_n^{\alpha}(x;k) \frac{(y/k)^{kn} z^n}{(\alpha+1)_{kn}}$$

$$= exp\{z[(y/k)^k - (x/k)^k]\} \sum_{n=0}^{\infty} Z_n^{\alpha}(y;k) \frac{(x/k)^{kn} z^n}{(\alpha+1)_{kn}}, \quad (4.6)$$

and on equating coefficients of Z^n in (4.6), the summation formula shall be led to our summation formula (4.1) .

4.2 Bilateral Generating function for $Y_n^c(x;k)$ Kanhouser Polynomials

Theorem 4.2

$$\sum_{n=0}^{\infty} Y_n^c(x;k)\, \zeta_n(y) t^n$$

$$= (1-t)^{-(\alpha+1)/k}\ exp\{1-(1-t)^{-1/k}\}$$

$$\cdot\ G\left[x(1-t)^{-1/k}\ yt/(1-t)\right], \quad (4.7)$$

where

$$G[x.t] = \sum_{n=0}^{\infty} \lambda_n\, Y_n^c(x;k) t^n, \quad (4.8)$$

The $\lambda_n \neq 0$ are arbitrary constants and $\zeta_n(y)$ is a polynomial of degree n is y given by

$$\zeta_n(y) = \sum_{r=0}^{n} \binom{n}{r} \lambda_r \quad . \tag{4.9}$$

The following result will be required in further analysis,

$$\sum_{n=0}^{\infty} \binom{m+n}{n} Y_{m+n}^{\alpha}(x;k)$$

$$=(1-t)^{-(\alpha+mk+1)/k} \; exp\{x[1-(1-t)^{-1/k}]\} \; Y_m^{\alpha}\left(x(1-t)^{-1/k};k\right), \tag{4.10}$$

where m $\geq$ 0 is any integer.

Proof

Substituting for the coefficients $\zeta_n(y)$ from (4.9) on the left –hand side of (4.7) , it is found that

$$\sum_{n=0}^{\infty} Y_n^c(x;k)\, \zeta_n(y) t^n = \sum_{n=0}^{\infty} Y_n^c(x;k)\, t^n \sum_{r=0}^{n} \binom{n}{r} \lambda_r y^r$$

$$= \sum_{r=0}^{\infty} \lambda_r (yt)^r \sum_{r=0}^{n} \binom{n+r}{r} Y_{n+r}^c(x;k) t^n$$

$$= (1-t)^{-(\alpha+1)/k} \; exp\{x[1-(1-t)^{-1/k}]\}$$

$$\cdot \sum_{r=0}^{\infty} \lambda_r \, Y_r^{\alpha}\left(x(1-t)^{-1/k};k\right) \left(yt/(1-t)\right)^r, \tag{4.11}$$

By applying (4.10) and formula (4.7) would follow if interpret this last expression by means of (4.8).

Theorem 4.3

$$\sum_{n=0}^{\infty} Y_{n+m}^c(x;k)\, \zeta_n(y;z) t^n$$

$$= (1-t)^{-m-(\alpha+1)/k} exp\left(x[1-(1-t)^{-1/k}]\right)$$

$$F\left[x(1-t)^{-1/k}; y; zt^q/(1-t)^q\right], \tag{4.11}$$

where

$$F[x; y; t] = \sum_{n=0}^{\infty} \lambda_n \, Y_{m+qn}^{\alpha}(x; k) \sigma_n(y) t^n, \tag{4.12}$$

and

$$\zeta_n(y; z) = \sum_{n=0}^{[n/q]} \binom{m+n}{n-qr} \lambda_r \sigma_r(y) z^r. \tag{4.13}$$

Proof

The following known result is required.

$$\sum_{n=0}^{\infty} \binom{m+n}{n} Y_{m+n}^{\alpha}(x; k) t^n$$

$$= (1-t)^{-m-(\alpha+1)/k} \exp\left(x\left[1-(1-t)^{-1/k}\right]\right) Y_m^{\alpha}\left(x(1-t)^{-1/k}; k\right). \tag{4.14}$$

where m is an arbitrary nonnegative integer, and (by definition) $\alpha > -1$ and k = 1, 2 , 3 ,…, If we substituting for the coefficients $\zeta_n(y; z)$ from (4.13) into the left –hand side (4.11) , we find that

$$\sum_{n=0}^{\infty} Y_{n+m}^{c}(x; k) \, \zeta_n(y; z) t^n$$

$$= \sum_{n=0}^{\infty} Y_{n+m}^{c}(x; k) \, t^n \sum_{r=0}^{n/q} \binom{m+n}{n-qr} \lambda_r \sigma_r(y) z^r$$

$$= \sum_{r=0}^{\infty} \lambda_r \sigma_r(y) z^r \, t^{qr} \sum_{n=0}^{\infty} \binom{M+n}{n} Y_{M+n}^{\alpha}(x; k) \, t^n,$$

where for convenience, M = m + qr, r = 0, 1, 2 ,… .

The inner series can now be summed by applying generating relation (4.14) with m replaced by M, and the bilateral generating function (4.11) would follow if it is interpreted by the resulting expression by means of (4.12).

Theorem 4.4

$$\sum_{n=0}^{\infty} Y_{n+m}^{\alpha-kn}(x;k)\,\zeta_n(y;z)t^n \tag{4.15}$$

$$= (1+t)^{-1+(\alpha+1)/k} exp\big(x\big[1-(1+t)^{1/k}\big]\big)$$

$$\cdot\, G\big[x(1+t)^{1/k}; y; zt^q/(1+t)^q\big],$$

where

$$G[x;y;t] = \sum_{n=0}^{\infty} \lambda_n\, Y_{m+qn}^{\alpha-kqn}(x;k)\sigma_n(y)t^n\,,$$

Proof

Can be proven similarly by appealing to

$$\sum_{n=0}^{\infty} \binom{m+n}{n} Y_{m+n}^{\alpha-kn}(x;k)t^n = (1+t)^{-1+(\alpha+1)/k} \tag{4.16}$$

$$\cdot\, \exp\big(x\,\big[1-(1+t)^{1/k}\big]\big)\, Y_m^{\alpha}\big(x(1+t)^{1/k};k\big),$$

in place of

$$\sum_{n=0}^{\infty} \binom{m+n}{n} Y_{m+n}^{\alpha}(x;k)t^n = (1-t)^{-m-(\alpha+1)/k} \tag{4.17}$$

$$\cdot\, \exp\big(x\big[1-(1-t)^{-1/k}\big]\big)\, Y_m^{\alpha}\big(x(1-t)^{-1/k};k\big).$$

CHAPTER 5

BIORTHOGONAL POLYNOMIALS SUGGESTED BY THE JACOBI POLYNOMIALS

In this chapter, another pair of biorthogonal polynomials that are suggested by the classical Jacobi polynomials will be introduced(Madhekar & Thakare, 1967). Let $\alpha > -1, \beta > -1$ and $J_n(\alpha,\beta,k\,;x)$ and $K_n(\alpha,\beta,k\,;x)$, $n = 0, 1, 2, \ldots$ be respectively the polynomials of degree n in x^k and x is real, k is positive integer such that these two polynomial sets satisfy biorthogonality conditions with respect to the weight function $(1-x)^{\alpha}(1+x)^{\beta}$, namely

$$\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\beta}J_n(\alpha,\beta,k\,;x)x^i dx$$

$$= \begin{cases} 0 & , \quad i = 0,1,\ldots,n-1; \\ \neq 0 & , \quad i = n\,; \end{cases} \tag{5.1}$$

and

$$\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\beta}K_n(\alpha,\beta,k\,;x)x^{ki} dx$$

$$= \begin{cases} 0 & , \quad i = 0,1,\ldots,n-1; \\ \neq 0 & , \quad i = n\,. \end{cases} \tag{5.2}$$

It follows from (5.1) and (5.2) that

$$\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\beta}J_n(\alpha,\beta,k\,;x)\,K_m(\alpha,\beta,k\,;x) dx$$

$$= \begin{cases} 0 & , \quad m,n = 0,1,\ldots, m \neq n; \\ \neq 0 & , \quad m = n; \end{cases} \tag{5.3}$$

And conversely, for k =1 both these sets are reduced to Jacobi polynomial sets .

There is a classical result which connects the classical Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ with the Laguerre polynomials $L_n^{(\alpha)}(x)$ in following manner.

$$\Gamma(1+\alpha+\beta+n)P_n^{(\alpha,\beta)}(x) = \int_0^\infty t^{\alpha+\beta+\mathrm{n}}\, e^{-t}\; L_n^{(\alpha)}\left(\frac{1-x}{2}t\right) dt. \qquad (5.4)$$

This result has made it possible to introduce, the first set from the pair of biorthogonal polynomials $J_n(\alpha,\beta,k;x)$ and $K_n(\alpha,\beta,k;x)$ that are suggested by the Jacobi polynomials.

Let define the first set $J_n(\alpha,\beta,\mathrm{k}\,;x)$ by

$$\Gamma(1+\alpha+\beta+n)J_n(\alpha,\beta,k\,;x) = \int_0^\infty t^{\alpha+\beta+\mathrm{n}}\, e^{-t}\; Z_n^{(\alpha)}\left(\frac{1-x}{2};k\right) dt\,, \qquad (5.5)$$

$$\text{for } \alpha+\beta\; >\; -1,\;\; n=0,1,2,\dots\,.$$

Using (3.5) on obtains by routine calculations

$$J_n(\alpha,\beta,k\,;x) \;=\; \frac{(1+\alpha)_{kn}}{n!}\sum_{j=0}^{n}(-1)^j\binom{n}{J}\frac{(1+\beta+\alpha+n)_{\mathrm{kj}}}{(1+\alpha)_{\mathrm{kj}}}\left(\frac{1-x}{2}\right)^{kj}. \qquad (5.6)$$

In fact $J_n(\alpha,\beta,\mathrm{k}\,;x)$ has the following hypergeometric form

$$J_n(\alpha,\beta,k\,;x) = \frac{(1+\alpha)_{kn}}{n!}\;\; {}_{k+1}F_k\left[\begin{matrix}-n,\Delta(k,1+\alpha+\beta); \\ \\ \Delta(k,1+\alpha);\end{matrix}\quad (\frac{1-x}{2})^k\right];\quad (5.7)$$

where $\Delta(m,\delta)$ stands for the sequence of m parameters

$$\frac{\delta}{m},\frac{\delta+1}{m}\,,\dots,\frac{\delta+m+1}{m}\,,m\geq 1.$$

The polynomials $\{J_n(\alpha,\beta,k\,;x)\}$ were first introduced Chai and Carlitz published the proof of their biortjogonality to x^i (i.e., of type (5.1) with respect to $x^{\,\alpha}(1+x)^\beta$ on (0 , 1). Chai proposal was on (0 , 1) instead of our (-1 , 1) . This also reminds one of the transition of the classical Jacobi polynomials first denoted by $F_n(\alpha\,,\beta,k;x)$ and orthogonal with respect to the weight function $x^{\,\alpha}(1+x)^\beta$ on (0 ,1) to that of Szego's standardized Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ which are orthogonal with respect to the weight function $(1-x)^{\alpha}(1+x)^{\beta}$ over the interval $(-1, 1)$.

The second set $K_n(\alpha,\beta,\mathrm{k}\,;x)$ is retroduced in the form of the following explicit representation

$$K_n(\alpha,\beta,\mathrm{k}\,;x)= \sum_{r=0}^{n}\sum_{s=0}^{r}(-1)^{r+s}\binom{r}{s}\frac{(1+\beta)_n}{n!\,r!\,(1+\beta)_{n-r}}\left(\frac{s+\alpha+1}{k}\right)_n\left(\frac{x-1}{k}\right)^r. \quad (5.8)$$

For k = 1, both $K_n(\alpha,\beta,\mathrm{k}\,;x)$ and $J_n(\alpha,\beta,\mathrm{k}\,;x)$ get reduced to the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$.

It is easy to observe that

$$\begin{cases}\lim\limits_{\beta\to\infty} J_n\left(\alpha,\beta,\mathrm{k}\,;1-\dfrac{2x}{\beta}\right)= Z_n^c(x;k)\\ \lim\limits_{\beta\to\infty} K_n\left(\alpha,\beta,\mathrm{k}\,;;1-\dfrac{2x}{\beta}\right)= Y_n^c(x;k)\end{cases} \quad (5.9)$$

For k =1, each of (5.9) gives well known connection between the Jacobi polynomials and Laguerre polynomials.

5.1 Biorthogonality

Employing the explicit formulas (5.6) and (5.8) we will show that the pair polynomials$K_n\left(\alpha,\beta,k\,;;1-\frac{2x}{\beta}\right)$ and $J_n\left(\alpha,\beta,k\,;1-\frac{2x}{\beta}\right)$ satisfies the biorthogonality condition(5.3) in fact ,we have

$$I_{n,m} = \int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta} J_n(\alpha,\beta,\mathrm{k};x)\, K_m(\alpha,\beta,\mathrm{k};x)dx$$

$$= \frac{\Gamma(1+\alpha+kn)\Gamma(1+\beta+m)}{2^m n!\, m!\, \Gamma(1+\alpha+\beta+n)} \sum_{j=0}^{n} (-1)^j \binom{n}{J} \frac{\Gamma(1+\alpha+\beta+n+kj)}{2^{kj}\Gamma(1+\alpha+kj)}$$

$$\times \qquad \sum_{r=0}^{n}\sum_{s=0}^{r} (-1)^s \binom{r}{s} \left(\frac{s+\alpha+1}{k}\right)_m \frac{1}{\mathrm{r}!\,\Gamma(1+\beta+m-r)}$$

$$\times \int_{-1}^{1} (1-x)^{\alpha+\mathrm{kj}+\mathrm{r}}(1+x)^{\beta+m-r}\, dx$$

$$= 2^{1+\alpha+\beta} \frac{\Gamma(1+\alpha+kn)\Gamma(1+\beta+m)}{n!\, m!\, \Gamma(1+\alpha+\beta+n)} \sum_{j=0}^{n} (-1)^j \binom{n}{J} \frac{\Gamma(1+\alpha+\beta+n+kj)}{\Gamma(2+\alpha+\beta+m+kj)}$$

$$\times \sum_{r=0}^{m} \binom{\alpha+k\mathrm{j}+r}{r} \sum_{s=0}^{r} (-1)^s \binom{r}{s} \left(\frac{s+\alpha+1}{k}\right)_m .$$

Recall the following result of Carlitz:

$$\left(\frac{s+\alpha+1}{k}\right)_n = \sum_{r=0}^{m} \binom{-x+r-1}{r} \sum_{s=0}^{r} (-1)^s \binom{r}{s} \left(\frac{s+\alpha+1}{k}\right)_n .$$

Using this,

$$\mathrm{I}_{\mathrm{n,m}} = 2^{1+\alpha+\beta} \frac{\Gamma(1+\alpha+kn)\Gamma(1+\beta+m)}{n!\, m!\, \Gamma(1+\alpha+\beta+n)}$$

$$\times \sum_{j=0}^{n} (-1)^j \binom{n}{J} (-\mathrm{j})_{\mathrm{m}} \frac{(1+\beta+\alpha)_{n+k\mathrm{j}}}{(1+\beta+\alpha)_{m+kj+1}}$$

$$= 2^{1+\alpha+\beta} \frac{\Gamma(1+\alpha+kn)\Gamma(1+\beta+m)}{n!\, \Gamma(1+\alpha+\beta+n)} (-1)^m \binom{n}{m}$$

$$\times \sum_{j=0}^{n} (-1)^j \binom{n-m}{J-m} \frac{(1+\beta+\alpha)_{n+kj}}{(1+\beta+\alpha)_{m+k\mathrm{j}+1}}$$

$$= 2^{1+\alpha+\beta} \frac{\Gamma(1+\alpha+kn)\Gamma(1+\beta+m)}{n!\, \Gamma(1+\alpha+\beta+n)} \binom{n}{m}$$

$$\times \sum_{j=0}^{n-m} (-1)^j \binom{n-m}{J} \frac{(1+\beta+\alpha)_{n+km+kj}}{(1+\beta+\alpha)_{m+km+k\mathrm{j}+1}}$$

$$= 2^{1+\alpha+\beta} \frac{\Gamma(1+\alpha+kn)\Gamma(1+\beta+m)}{n!\,\Gamma(1+\alpha+\beta+n)} \binom{n}{m}$$

$$\times \sum_{j=0}^{n-m} (-1)^j \binom{n-m}{J} D^{n-m-1} x^{\alpha+\beta+n+km+kj} |_{x=1}$$

$$= 2^{1+\alpha+\beta} \frac{\Gamma(1+\alpha+kn)\Gamma(1+\beta+m)}{n!\,\Gamma(1+\alpha+\beta+n)} \binom{n}{m}$$

$$\times D^{n-m-1} x^{\alpha+\beta+n+km} \ (1-x^k)^{n-m}|_{x=1}$$

which is 0 for $n \neq m$ and nonzero for $n = m$. In particular,

$$\mathrm{I}_{n,n} = 2^{1+\alpha+\beta} \frac{\Gamma(1+\alpha+kn)\Gamma(1+\beta+n)}{n!\,\Gamma(1+\alpha+\beta+n)\Gamma(1+\beta+\alpha+n+kn)} \ .$$

Conclusions

In this thesis , definitions and basic properties of konahser polynomials $Y_n^c(x;k)$ and $Z_n^c(x;k,$ in x and x^krespectively, are investigated , such as biorthogonality , recurrence relations and differential equations .

Moreover, some generalizations of biorthogonal polynomials was obtained, and based on these generalizations many new ideas can be applied .

Finally , biorthogonal polynomials suggested by the Jacobi polynomials , are given and the biorthogonality are proven.

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